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### ABSTRACT

The deviation of a system function from its nominal value due to random variations or the inherent uncertainty of the system parameters is examined. Bounds on various probabilistic measures of this deviation are discussed and examples are given. A comparison is made with the deterministic multiparameter sensitivity function.

### I. INTRODUCTION

The significance of parameter variation in linear networks and systems is well known for the case of deterministic models. [1-7]. In many cases, however, it is more appropriate to make use of a probabilistic approach in studying the sensitivity problem. This is particularly true when the number of parameters in the network is not small.

Considerable interest has been shown in worst-case variation as a sensitivity measure of a system, and procedures have been given for maximum (minimum) search by computer [8]. An alternative approach to this problem has been that of deriving bounds of the variations [9]. However, the bounds derived so far tend to be pessimistic [8]. We give here a simple way of calculating the approximate worst-case performance when the variation of parameters is not large. A comparison is made of this variation with the results obtained using the probabilistic approach.

### II. SINGLE PARAMETER CASE

Consider the system function which describes a linear network,  $H(\omega, x)$  where  $\omega$  is frequency and  $x$  is an element in the network. It is well known that  $H$  is a bilinear function of  $x$ , i.e.,

$$H(\omega, x) = \frac{A(\omega) + x B(\omega)}{C(\omega) + x D(\omega)} \quad (1)$$

Classical sensitivity  $S_x^H$ , defined by

$$S_x^H = \frac{\partial \ln H}{\partial \ln x} = \frac{x}{H} \frac{\partial H}{\partial x} \quad (2)$$

has been used in the study of the change of system performance with respect to parameter variation. The real and imaginary parts of  $S_x^H$  are related separately to the amplitude and phase variations of  $H$  caused by a small change  $\Delta x$  in  $x$ . The first-order effects are given by

$$\frac{\Delta |H|}{|H|} \sim \text{Re } S_x^H \frac{\Delta x}{x} \quad (3)$$

\* Presented at the Fifth Annual Allerton Conference on Circuit and System Theory, October 4-6, 1967. To be published in the Proceedings of the Conference.

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$$\Delta \arg H \approx \operatorname{Im} S_x^H \frac{\Delta x}{x} \quad (4)$$

where  $|H|$  and  $\Delta|H|$  and  $\Delta \arg H$  denote respectively the magnitude of  $H$ , the change in  $|H|$ , and the change in the argument of  $H$ .

Traditionally, the problem of system sensitivity and parameter variation has been treated by using a deterministic approach by assuming the variation  $\Delta x$  is a fixed quantity. However, strictly speaking, the element values in a system are only known probabilistically in view of inherent uncertainties. In addition, our knowledge of the variation of the variation of these element values due to aging, environment changes, etc. is conditioned by a certain degree of incompleteness. It is therefore more reasonable to take  $x$  as a random variable whose mean  $\bar{x}$  is regarded as its nominal value and whose variance  $\sigma_x^2$  gives a measure of its inherent inaccuracy.

At a fixed  $\omega$ , the expected value of  $H(\omega, x)$  as given in Eq. 1 is given by

$$E\{H(\omega, x)\} = \int \frac{A(\omega) + xB(\omega)}{C(\omega) + xD(\omega)} p(x) dx \quad (5)$$

where  $E$  denotes the expectation, and  $p(x)$  is the probability density of  $x$ . Since one normally takes

$$H(\omega, \bar{x}) = \frac{A(\omega) + \bar{x}B(\omega)}{C(\omega) + \bar{x}D(\omega)} \quad (6)$$

as the nominal  $H$ , it is of interest to compare the two nominal values of  $H$  as given by Eqs. 5 and 6.

We may write  $x$  as  $\bar{x} + \Delta$ , where the random variable  $\Delta$  has zero mean and variance  $\sigma_x^2$ . If  $\sigma_x$  is small,  $H(\omega, x)$  may be expanded in a power series

$$H(\omega, x) = H(\omega, \bar{x}) + \frac{\partial H}{\partial x} \Delta + \frac{1}{2} \frac{\partial^2 H}{\partial x^2} \Delta^2 + \dots \quad (7)$$

and the first few terms can be taken as a good approximation. Since  $E\{\Delta\} = 0$ , it can be seen that the difference between  $E\{H(\omega, x)\}$  and  $H(\omega, \bar{x})$  is a second-order effect.

This result can also be seen in terms of a graphical representation. The bilinear Eq. 1 maps the real  $x$  line into a circle for a fixed  $\omega$  as illustrated in Fig. 1(a). Suppose  $x$  deviates from its nominal value  $\bar{x}$  by a fixed  $\delta$ . Then  $H(\omega, \bar{x} + \delta)$  should be on the circle as shown in the slightly expanded illustration about  $H(\omega, \bar{x})$  of Fig. 1(b). The approximation,  $H(\omega, \bar{x}) + \frac{\partial H}{\partial x} \delta$ , obtained by taking only the first two terms in the series expansion, Eq. 7, lies on the tangent to the circle at  $H(\omega, \bar{x})$ . Similarly,  $H(\omega, \bar{x} - \delta)$  is on the circle and  $H(\omega, \bar{x}) - \frac{\partial H}{\partial x} \delta$  is on the tangent. Since the approximation

lies on a straight line and since the random deviation is assumed to have zero mean, the resulting approximation to  $E\{H(\omega, x)\}$  coincides with  $H(\omega, x)$ . The actual  $E\{H(\omega, x)\}$  would lie somewhere inside the circle as shown. By taking more terms in the power series, the approximation to the expected value of  $H$  would approach its true value.

If the exact distribution of  $x$  is known, or a reasonable one can be assumed, then  $E\{H(\omega, x)\}$  can be determined, though in most cases numerical calculation by a computer is required. For example, if  $x$  is gaussian with mean  $\bar{x}$  and variance  $\sigma_x^2$ , Eq. 5 can be written as<sup>1</sup>

$$E\{H(\omega, x)\} = \frac{B}{D} + \left(\frac{E}{D} - \frac{BF}{D^2}\right) \frac{1}{\sigma_x} \sqrt{\frac{\pi}{2}} W\left(\frac{-F}{\sqrt{2}\sigma_x D}\right) \quad (8)$$

where  $E = A + \bar{x}B$ ,  $F = C + \bar{x}D$  and the function  $W(z)$  is related to the complementary error function by [10]  $W(z) = e^{-z^2} \text{erfc}(-iz)$ . If  $x$  is uniformly distributed between  $x \pm \sqrt{3} \sigma_x$ , the result is

$$E\{H(\omega, x)\} = \frac{B}{D} + \frac{ED - BF}{2\sqrt{3}\sigma_x D} \ln \frac{F + \sqrt{3}\sigma_x D}{F - \sqrt{3}\sigma_x D} \quad (9)$$

Provided  $C/D$  is not real.

### III. MULTIPLE PARAMETER CASE

Let  $x_1$  and  $x_2$  be two elements in a linear network. The system function  $H(\omega, x_1, x_2)$  is then bilinear in each of  $x_1$  and  $x_2$ . We take  $x_1$  and  $x_2$  to be independent random variables with means  $\bar{x}_1$  and  $\bar{x}_2$  and variances  $\sigma_1^2$  and  $\sigma_2^2$ . Again,  $H(\omega, \bar{x}_1, \bar{x}_2)$  is usually taken to be the nominal  $H$  and it can be argued as before that  $H(\omega, \bar{x}_1, \bar{x}_2)$  is a good approximation of the actual mean  $E\{H(\omega, x_1, x_2)\}$ .

At each  $\omega$ , and for each fixed  $x_2$ , the locus of  $H$  as a function of  $x_1$  is a circle, and for each  $x_1$ , the locus of  $H$  as a function of  $x_2$  is also a circle. Suppose the value of  $x_1$  is to vary between  $\bar{x}_1 \pm \delta_1$  and the range of  $x_2$  is  $\bar{x}_2 \pm \delta_2$ , then the variation of  $H(\omega, x_1, x_2)$  is as illustrated in Fig. 2. The value of  $H$  is then limited to the region bounded by the four dashed circular arcs. If none of these four arcs is tangent to circles with center at the origin, or to straight lines passing through the origin, it is seen that the extremes of  $|H|$  and  $\arg H$  can occur only at the "corner points," a, b, c, or d. Even when this is not true because some of the arcs are tangent to circles with center at origin or to straight

1. For simplicity, the argument  $\omega$  is sometimes omitted.

lines through the origin, the extreme values evaluated at the corners are still very good approximations to the true extremes. The above argument can be extended to situations with more than two parameters. Figure 3 illustrates a 3-parameter case.

The significance of the problem of worst-case variation has been recognized by many authors. Most of the results in the past have concentrated along two lines. The first is to device bounds which usually turn out to be rather pessimistic. The second is to actually search for the extremes in the parameter space and usually requires considerable computational effort. The observation made above can be used to determine approximately the worst case variation of  $H$  by simply evaluating  $H$  at all of the corner points and comparing the results. Thus for a network of  $n$  parameters,  $x_i$  with range  $\bar{x}_i \pm \delta_i$ ,  $i = 1, 2, \dots, n$ , one needs to calculate  $H$  for each of the  $2^n$  combinations of the parameter values,  $x_i = \bar{x}_i \pm \delta_i$ . The extremums obtained in this manner can be taken as approximations of the actually extremums.

As an illustration, consider the third-order Chebyshev low-pass filter shown in Fig. 4. The sensitivity problem of this network has been studied by Kelly [8] who calculated the actual maximum and minimum of the amplitude characteristic. The result of calculation based upon the approach outlined above is shown in Fig. 5 for  $\pm 5\%$  and  $\pm 10\%$  variations of all elements. Also shown is the actual maximum and minimum as calculated by Kelly. For this order of parameter variation, it is seen that our method gives reasonably good approximations except near the cut off frequency where the change is fast. However, if the range of variation is increased, one should expect that our approximation procedures will not give reliable results.

#### IV. MULTIPARAMETER SENSITIVITY AND STATISTICAL VARIATION

Let  $x_i$ ,  $i = 1, 2, \dots, n$  be element values. The first order approximation to the variation in the system function  $H(\omega, x_1, x_2, \dots, x_n)$  are related to the real and imaginary parts of the sensitivities through the relationships

$$\frac{\Delta |H|}{|H|} \approx \sum_{i=1}^n (\operatorname{Re} S_{x_i}^H) \frac{\Delta x_i}{x_i} \quad (10)$$

and

$$\Delta \arg H \approx \sum_{i=1}^n (\operatorname{Im} S_{x_i}^H) \frac{\Delta x_i}{x_i} \quad (11)$$

where the sensitivities  $S_{x_i}^H$  are defined in the usual manner, i.e.,

$$S_{x_i}^H = \frac{\partial H}{\partial x_i} \frac{x_i}{H} \quad (12)$$

and the  $\Delta x_i$ 's are the deviations from their nominal values  $x_i$ . We take the  $x_i$ 's to be independent random variables with mean  $\bar{x}_i$  and variance  $\sigma_i^2$ . Thus

$$x_i = \bar{x}_i + \delta_i \quad (13)$$

with

$$E\{\delta_i\} = 0, \quad E\{\delta_i^2\} = \sigma_i^2 \quad (14)$$

Equations 10 and 11 may be written as\*

$$\frac{\Delta |H|}{|H|} \approx \sum_{i=1}^n (\operatorname{Re} S_{x_i}^H) \frac{\delta_i}{x_i} \quad (15)$$

and

$$\Delta \arg H \approx \sum_{i=1}^n (\operatorname{Im} S_{x_i}^H) \frac{\delta_i}{x_i} \quad (16)$$

with  $|H| = |H(\omega, x_1, x_2, \dots, x_n)|$ . In view of Eqs. 13 and 14, we have

$$E\left\{\frac{\Delta |H|}{|H|}\right\} = 0 \quad (17)$$

$$E\{\Delta \arg H\} = 0 \quad (18)$$

$$E\left\{\left[\frac{\Delta |H|}{|H|}\right]^2\right\} = \sum_{i=1}^n (\operatorname{Re} S_{x_i}^H)^2 \sigma_i^2 \quad (19)$$

and

$$E\{[\Delta \arg H]^2\} = \sum_{i=1}^n (\operatorname{Im} S_{x_i}^H)^2 \sigma_i^2 \quad (20)$$

If the range of  $\delta_i$  is restricted to  $\pm \Delta_i$ , then it is clear that the extrema of  $\Delta |H|/|H|$  are given by

$$\max \frac{\Delta |H|}{|H|} = \sum_{i=1}^n |\operatorname{Re} S_{x_i}^H| \frac{\Delta_i}{x_i} \quad (21)$$

and

$$\min \frac{\Delta |H|}{|H|} = - \sum_{i=1}^n |\operatorname{Re} S_{x_i}^H| \frac{\Delta_i}{x_i} \quad (22)$$

One may take  $\pm k\sigma_i$ , with a proper choice of  $k$ , as the range of the variation of parameter<sup>2</sup>  $x_i$ . Then the range of variation using the extrema is

$$\text{Range}_I = 2k \sum_{i=1}^n |\operatorname{Re} S_{x_i}^H| \frac{\sigma_i}{x_i} \quad (23)$$

2 That is, if  $k$  is taken as 3, then the element will be within this range with probability better than 0.997.

\*In the discussion to follow, we replace  $\approx$  by  $=$  rather than discuss the precise meaning of the approximation.

and the range using  $k \sigma \frac{\Delta|H|}{|H|}$ , where  $\sigma^2 \frac{\Delta|H|}{|H|}$  is the variance of  $\Delta|H|/|H|$ , and is

$$\text{Range}_{II} = 2k \sqrt{\sum_{i=1}^n (\text{Re } S_{x_i}^H)^2 \frac{\sigma_i^2}{x_i^2}}$$

That  $\text{Range}_I \geq \text{Range}_{II}$  follows from an elementary inequality. In fact, for most cases, one will find  $\text{Range}_{II}$  to be considerably smaller than  $\text{Range}_I$ .

The third-order Chebyshev filter is used again as an illustration. The sum of the absolute value  $\text{Re } S_{x_i}^H$  is calculated to be 1.3390 at  $\omega/2\pi = 0.1$ , 1.4013 at  $\omega/2\pi = 0.15$ , and 2.8755 at  $\omega/2\pi = 0.2$ , where as the square root of the sum of the squares of the real parts of the individual sensitivities are 0.7756, 0.7310, and 1.6730 at these frequencies. That is, for this example, the extreme range of variation  $\text{Range}_I$  is almost twice as large as a meaningful measure should be. Figure 6 shows the ranges as determined by the extrema and as determined by the variances of  $\Delta|H|/|H|$  at  $\omega = 1.0$  and  $\omega = 1.5$ , using  $k = 3$ . Also sketched in the figure are the probability density functions.

When the distributions of the elements are not gaussian, one can not draw the definitive result as given above. But if the number of terms in Eq. 15 is not too small and if no term in the sum clearly dominates, then the final distribution of  $\Delta|H|/|H|$  will still be approximately gaussian [11]. Suppose the elements in the example of the Chebyshev filter are uniformly distributed, then the probability density of  $\Delta|H|/|H|$  is that of the successive convolution of five rectangular shapes of proper width.

The result is also sketched in Fig. 6. If one takes  $\pm 3\sigma$  as the range of variation, it is seen that with probability better than .92,  $\Delta|H|/|H|$  will be within the range when the elements are uniformly distributed.

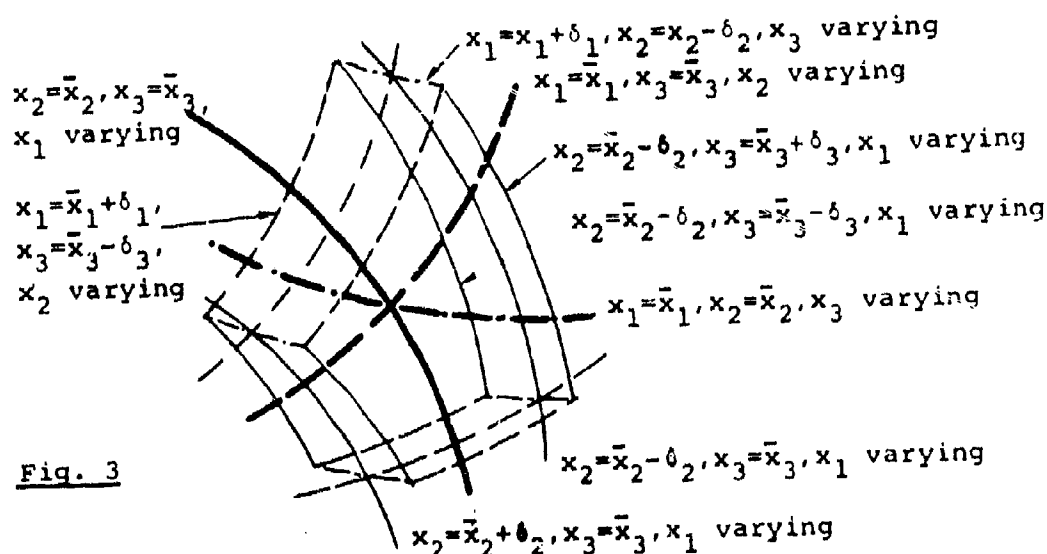
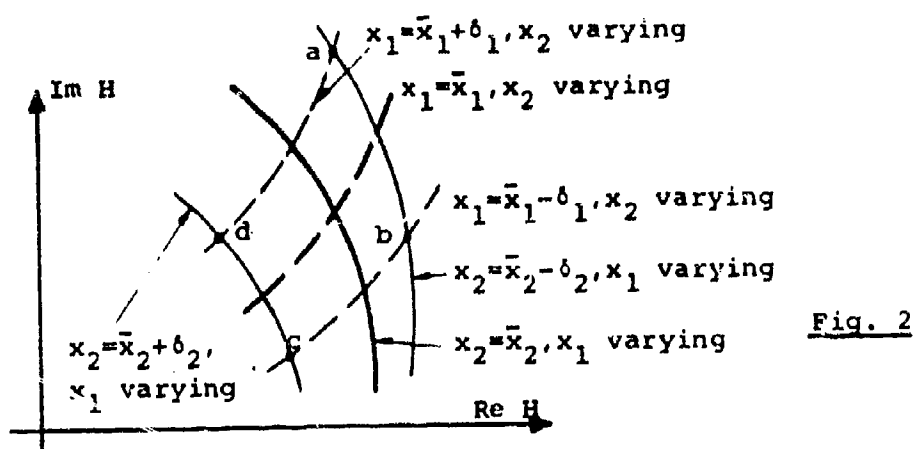
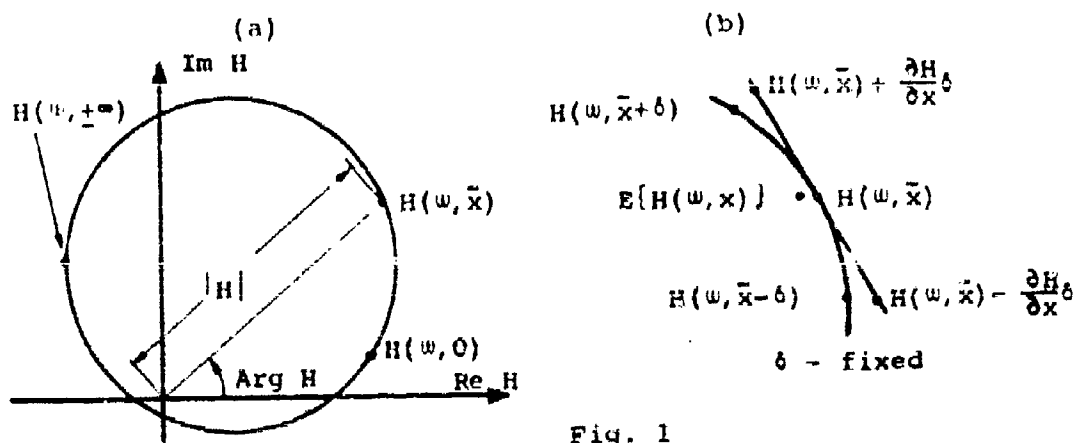
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# REFERENCES

1. Blostein, M.L., "On the Effect of Loss in Filter Networks," Proc. 3rd Allerton Conference on Circuit and System Theory, 1965; pp. 421-438.
2. Bihovski, M.L., "The Accuracy of Electric Network Analyzers" IZV an SSSR, Otd. Tekhn. Nauk, No. 8, 1948.
3. Cruz, J.B., Jr. and W. R. Perkins, "The Role of Sensitivity in the Design of Multivariable Linear Systems," Proc. NEC, 1954; pp. 742-745.
4. Cruz, J.B., Jr. and W. R. Perkins, "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Trans. on Automatic Control, vol. AC-9, pp. 216-223; July 1964.
5. Goldstein, A.J. and F. F. Kuo, "Multiparameter Sensitivity" IRE Trans. on Circuit Theory, vol. CT-8, No. 2, pp. 177-178; June 1961.
6. Horowitz, I.M., Synthesis of Feedback Systems, Academic Press, New York, 1963.
7. Lee, S.C., "On Multiparameter Sensitivity," Third Allerton Conf. on Circuit and System Theory, October 1965, pp. 407-420.
8. Kelly, Gerry L., "Nonlinear Programming and Multiparameter Sensitivity in Linear Time-Invariant Networks," Report R-301, Coordinated Science Laboratory, University of Illinois, June 1966.
9. Hakimi, S.L. and J. B. Cruz, Jr., "Measures of Sensitivity for Linear Systems with Large Multiple Parameter Variation" IRE WESCON Convention Record, Part 2, pp. 109-115; August 1960.
10. Abramowitz, M. and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965, pp. 297.
11. Papoulis, Athanasios, Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Co., 1965.





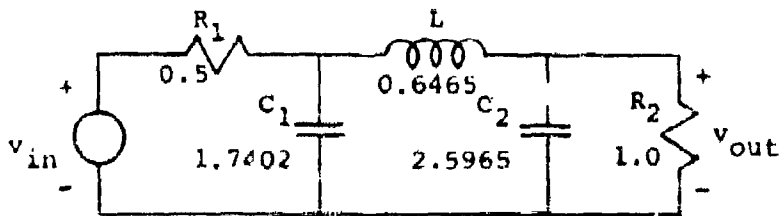


Fig. 4

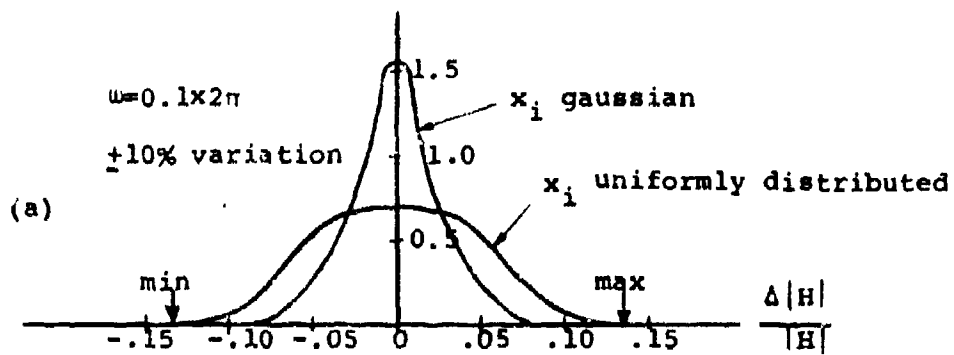
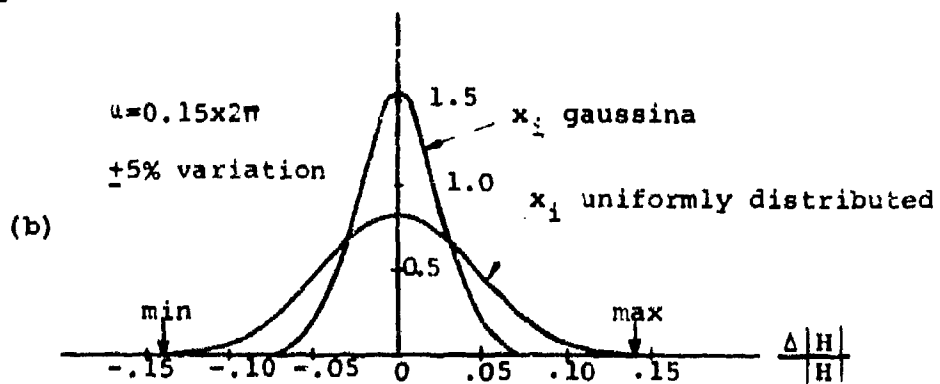


Fig. 6



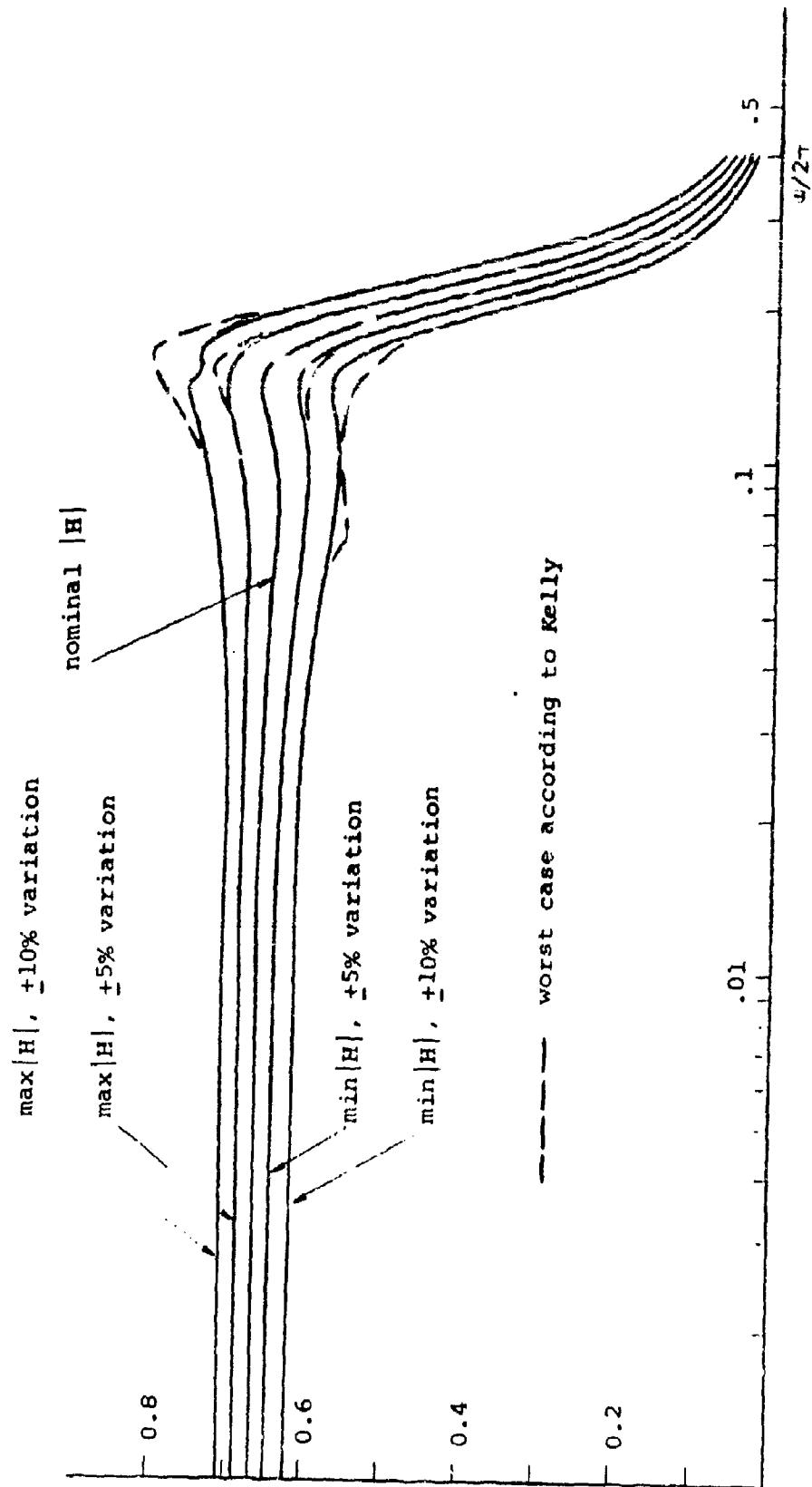


Fig. 5

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<p>13. ABSTRACT</p> <p>The deviation of a network function from its nominal value due to random variations or the inherent uncertainty of network parameters is studied in this paper.</p> <p>Consider the network function <math>H(\omega, x_1, x_2, \dots, x_n)</math> where <math>\omega</math> is the frequency variable and the <math>x</math>'s are the parameters. It is well known that <math>H</math> is bilinear in each of the <math>x</math>'s. Each <math>x_i</math> is a random variable with mean <math>\bar{x}_i</math> and variance <math>\sigma_i^2</math>. The mean <math>\bar{x}_i</math> is usually taken to be the nominal value and the variance is related to the tolerance. It is shown that the mean of <math>H</math>, <math>E[H]</math>, is approximately equal to <math>H(\omega, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)</math> the value of <math>H</math> when all the parameters take on their nominal values. For small <math>\sigma_i/\bar{x}_i</math>, the variances of <math>H</math> are shown to be related to the classical single parameter sensitivity <math>S_{x_i}^H</math>. When the <math>x</math>'s are Gaussian, the distribution of <math>H</math> can be determined. When the <math>x</math>'s are not Gaussian, various probabilistic bounds can be obtained by inequalities such as that of Chebyshev.</p> <p>This paper also presents a method of calculating approximately the deterministic worst case parameter variation. A comparison is made of this variations with the results obtained using the probabilistic approach, and it is shown that the conventional deterministic worst case design criterion is in general too pessimistic, especially when the number of parameters is not small.</p>			

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	ROLE	WT	ROLE	WT	ROLE	WT
Network Sensitivity Multiparameter variation Tolerance Limit Random variation						